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Master's Thesis

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Associativity in Stable homotopy theory

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Abstract

In this thesis, we construct an obstruction theory for $\mathbb{A}_n$-algebra structures in stable ∞-categories. We use this to show that the spectrum $\mathbb{S}/4$ admits an $\mathbb{A}_5$-multiplication.

Contents

1 Introduction

In higher algebra, a classic problem is what multiplicative structures there exists on quotients of the sphere spectrum. In the discrete analogue, all abelian groups $\mathbb{Z}/n\mathbb{Z}$ admit unique commutative multiplications, so one would expect the spectra $\mathbb{S}/n$ admit $\mathbb{E}_\infty$-ring structures, which is the higher analogue of commutative multiplications. However what algebraic structures cofibers admit in higher algebra are much more complicated, depending highly on what number the cofiber is taken of. The following results summarises our current knowledge of this problem:[1]

1. For a prime p, the Moore spectrum $\mathbb{S}/p$ admits an $\mathbb{A}_{p-1}$-algebra structure but not an $\mathbb{A}_p$-algebra structure.
2. For $q \geq \frac{3}{2}(n+1)$, the Moore spectrum $\mathbb{S}/2^q$ admits an $\mathbb{E}_n$-algebra structure. For p odd and $q \geq n+1$, the Moore spectrum $\mathbb{S}/p^q$ admits an $\mathbb{E}_n$-algebra structure.
3. The Moore spectrum $\mathbb{S}/4$ admits a $\mathbb{A}_4$-algebra structure, but not an $\mathbb{E}_2$-algebra structure.

While the first two results are quite strong, it is still an open question whether $\mathbb{S}/4$ admits an $\mathbb{E}_1$-algebra structure. The goal of the thesis is to give an improvement of the current result:

Theorem A (Corollary 6.9)**.** *The Moore spectrum $\mathbb{S}/4$ admits an $\mathbb{A}_5$-algebra structure.*

In Section 2 we introduce and define $\mathbb{A}_\mathrm{n}$-algebras. In Section 3 we construct an obstruction theory for $\mathbb{A}_\mathrm{n}$-algebra structures on a object in a stable ∞-category. Such an obstruction theory is not new, with the first exposition given by Alan Robinson in [Rob89] for $\mathbb{A}_\mathrm{n}$-ring spectra. We will use a construction developed by Robert Burklund in [Bur22], which works in a general stable symmetric monoidal ∞-category.

Theorem B (Proposition 3.9)**.** *Given a symmetric monoidal stable ∞-category $\mathcal{C}$ and a map $r : X \to 1_\mathcal{C}$, there exists a sequence of inductively defined obstructions*

$$\theta_k \in [\Sigma^{2k-3} X^{\otimes k}, 1_\mathcal{C}/r],$$

such that the vanishing of $\theta_1, \ldots, \theta_n$ induces a $\mathbb{A}_\mathrm{n}$-algebra structure on $1_\mathcal{C}/r$ with unit given by the cofiber map.

In Section 4 we shortly introduce the stable ∞-category of $\mathbb{F}_2$-synthetic spectra $\mathrm{Syn}_{\mathbb{F}_2}$, which is a deformation of the ∞-category of spectra Sp, with a symmetric monoidal functor $\tau^{-1} : \mathrm{Syn}_{\mathbb{F}_2} \to \mathrm{Sp}$. There is a map to the unit in synthetic spectra

$$\Sigma^{0,2} \nu\mathbb{S} \xrightarrow{\tilde{4}} \nu\mathbb{S},$$

[1] Proofs of these statements can be found in [Ang08],[Bur22] and [Bha22] respectively.

such that $\tau^{-1}\tilde{4} = 4$. An $\mathbb{A}_{\mathrm{n}}$-algebra structure on $\nu\mathbb{S}/\tilde{4}$ then induces a $\mathbb{A}_{\mathrm{n}}$-algebra structure on $\mathbb{S}/4$. Working with $\nu\mathbb{S}/\tilde{4}$ instead of $\mathbb{S}/4$, let us keep track of Adams filtration of maps, which is useful for the next section. In Section 5, we establish a functoriality of our obstruction theory, so that a monoidal functor F, will send the obstruction θ_k on an $\mathbb{A}_{k-1}$-algebra A to the obstruction for the induced $\mathbb{A}_{k-1}$-algebra structure on $F(A)$. We then define an $\mathbb{E}_\infty$-ring R and a R-module ${}_{\tilde{4}}\nu\mathbb{S}$ along with a monoidal functor

$$\begin{aligned} \mathrm{RMod}_R(\mathrm{Syn}_{\mathbb{F}_2}) &\to \mathrm{Syn}_{\mathbb{F}_2} \\ {}_{\tilde{4}}\nu\mathbb{S} &\mapsto \nu\mathbb{S}/\tilde{4}. \end{aligned}$$

If ${}_{\tilde{4}}\nu\mathbb{S}$ admits a $\mathbb{A}_4$-multiplication, we show that the map sending the obstruction θ_5 for ${}_{\tilde{4}}\nu\mathbb{S}$ to the obstruction for $\nu\mathbb{S}/\tilde{4}$, is contractible, which implies $\nu\mathbb{S}/\tilde{4}$ has an $\mathbb{A}_5$-multiplication. In the last section Section 6, we show that ${}_{\tilde{4}}\nu\mathbb{S}$ admits an $\mathbb{A}_4$-multiplication, by tackling the universal case, given by the following theorem.

Theorem C (Theorem 6.1)**.** *Let $\mathcal{C}$ be a 2-local presentable symmetric monoidal stable ∞-category, $X \in \mathrm{Pic}(\mathcal{C})$ and $v : X \to \mathbf{1}_{\mathcal{C}}$ be a map to the unit. If the map $Q_1(v) : \Sigma X^{\otimes 2} \to \mathbf{1}_{\mathcal{C}}$ vanishes, then $1/v$ admits a homotopy associative multiplication.*

1.1 Notations and Conventions

We will use the setting and language of ∞-categories developed by Jacob Lurie in [Lur09] and [Lur17].

1.2 Future work

In Section 3, Lemma 3.7 only shows that the functor commutes with n-fold tensor products, and not $\mathbb{A}_{\mathrm{n}}$-monoidal, which require either a more detailed analysis of Day convolution, or an alternative proof technique. The alternative construction of the obstruction theory relies on Claim 3.12, which is not proved in this thesis. I hope to later give a proof of this result, as it is both independently interesting, and is used in the proof of Proposition 5.1.

In Section 6, Lemma 6.4 I hope to extend this lemma, to similar result for odd primes. This should further lead to new proofs, that $\mathbb{S}/p$ is $\mathbb{A}_{p-1}$-monoidal.

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2 $\mathbb{A}_n$-Operads

There are different notions of an associative algebra structure on a topological space with a unit map $e : * \to X$. A topological monoid consists of a map $\mu : X \times X \to X$, that is strictly associative and unital by satisfying the following identities.

$$\mu(\mu \times 1) = \mu(1 \times \mu)$$
$$\mu(e \times \mathrm{id}_X) = \mu(\mathrm{id}_X \times e) = \mathrm{id}_x .$$

Another choice is an associative H-space, which instead of requiring the maps to strictly agree, only requires them to be homotopic

$$[\mu(\mu \times 1)] = [\mu(1 \times \mu)]$$
$$[\mu(e \times \mathrm{id}_X)] = [\mu(\mathrm{id}_X \times e)] = [\mathrm{id}_X].$$

Further, an $\mathbb{A}_2$-algebra is informally an associative H-space together with a choice of homotopy. In homotopy theory, associative H-space structures are more natural, since the spaces of interest are only considered up to homotopy. Associative H-spaces however, does not have the same useful properties as topological monoids have.

Remark 2.1. The $\mathbb{A}_2$-operad is not coherent in the sense of [Lur17], which is used for given a well-defined tensor product on the module category of an algebra. $\diamond$

The problems with an associative H-space, can be seen with multiplication of four elements. Each way of ordering the multiplication, gives a point in the space of maps $\mathrm{Map}_{\mathbf{Top}}(X^{\times 4}, X)$. Since the different ordering of multiplications are homotopic, we can choose paths in the mapping space connecting the different points, as can be seen in the following diagram.

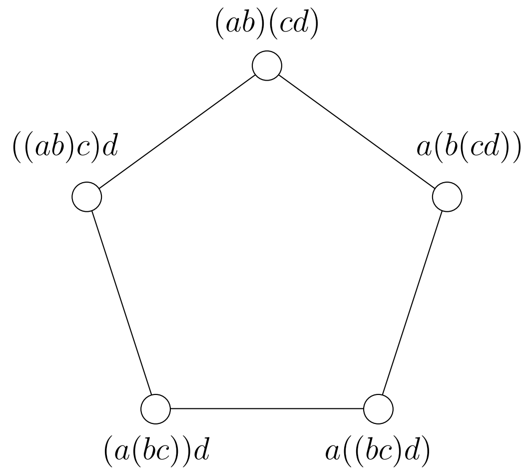


Figure 1: Space of multiplications of four elements.

From this picture we see that the space of multiplications of four elements, might not be contractible. If we want to have a unique multiplication of f elements up to contractible choice, we need a nulhomotopy of this loop. We can continue this inductively filling out homotopies in $\mathrm{Map}_{\mathbf{Top}}(X^{\times n}, X)$. We then get different associative multiplicative structures on X, depending on how many higher homotopies we require.

This leads to the definition of the $\mathbb{A}_n$-operads, first defined in [Sta63] and also described in [Bha22], involving the Stasheff-Associahedra polytopes K_n. In the language of planar ∞-operads $q : \mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{\mathrm{op}})$ introduced in [Lur17, Section 4.1.3][2], $\mathbb{A}_n$-algebras can be described simpler, by hiding the homotopies inside the notion of functors of ∞-categories.

Definition 2.2 ([Lur17] Definition 4.1.4.2 (Unital Version))**.** Let $q : \mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{\mathrm{op}})$ be a planar ∞-operad and let $0 \leq n \leq \infty$. An $\mathbb{A}_n$-algebra object of $\mathcal{C}$ is a functor $A : \mathrm{N}\left(\Delta^{\mathrm{op}}_{\leq n}\right) \to \mathcal{C}^{\circledast}$ with the following properties:

1. The following diagram commutes

$$\begin{tikzcd} & \mathcal{C}^{\circledast} \arrow[d, "q"] \\ \mathrm{N}\left(\Delta^{op}_{\leq n}\right) \arrow[ur, "A"] \arrow[r, hook, "i"'] & \mathrm{N}\left(\Delta^{op}\right). \end{tikzcd}$$

2. For every inert morphism $\alpha : [m'] \to [m]$ in Δ satisfying $1 \leq m' \leq m \leq n$, the induced map $A(\alpha) : A([m]) \to A([m'])$ is a q-coCartesian morphism of $\mathcal{C}^{\circledast}$.

[2]The ∞-category of planar ∞-operads $\mathrm{Op}^{\mathrm{pl}}_{\infty}$ is equivalent to the ∞-category $(\mathrm{Op}_{\infty})_{/\mathrm{Assoc}^{\otimes}}$ of fibrations of ∞-operads over $\mathrm{Assoc}^{\otimes}$. We use planar ∞-operads as the category Δ^{op} is easier to work with than $\mathrm{Assoc}^{\otimes}$.

We let $\mathrm{Alg}_{\mathbb{A}_n}(\mathcal{C})$ denote the full subcategory of

$$\mathrm{Fun}_{\mathrm{N}(\Delta^{\mathrm{op}})}(\mathrm{N}\left(\Delta^{\mathrm{op}}_{\leq n}\right), \mathcal{C}^{\circledast}) \times_{\mathrm{Fun}(\Delta^{\mathrm{op}}_{\leq n}, \Delta^{\mathrm{op}})} \{i\}$$

spanned by the unital $\mathbb{A}_n$-algebra objects of $\mathcal{C}$. We will refer to $\mathrm{Alg}_{\mathbb{A}_n}(\mathcal{C})$ as the ∞-category of $\mathbb{A}_n$-algebra objects of $\mathcal{C}$. $\diamond$

Remark 2.3. From this we see that $\mathbb{A}_n$-algebras are like $\mathbb{A}_\infty$-algebras, where only multiplication of up to n elements are well-defined. The ∞-category $\mathrm{Alg}_{\mathbb{A}_n}(\mathcal{C})$ can also be obtained as the ∞-category of algebras over an ∞-operad explained in [Lur17, Remark 4.1.4.8]. $\diamond$

3 Building $\mathbb{A}_n$-structure in stable categories

Constructing $\mathbb{A}_n$- and $\mathbb{A}_\infty$-algebras concretely is often practically impossible, as these structures involve a large structure of homotopies given by the Stasheff-polytopes K_n, which have complicated cellular structure. On the other hand, the Stasheff-polytopes K_n are homeomorphic to the disks D_{n-2} which are much simpler, leading to an obstruction theory for $\mathbb{A}_{\mathrm{n}}$-algebra structures.

We here give one construction based on filtered objects in a stable category. We also sketch a second construction, which however is not proven.

3.1 $\mathbb{A}_n$-monoids from algebras in filtered object

Definition 3.1. Let $\mathbb{Z}^{\mathrm{fil}}$ be the category with objects the integers and maps

$$\mathrm{Map}_{\mathbb{Z}^{\mathrm{fil}}}(n, m) = \begin{cases} * \text{ if } n \leq m \\ \emptyset \text{ if } n > m. \end{cases}$$

It is given the symmetric monoidal structure by addition on objects. $\diamond$

Definition 3.2. For any ∞-category $\mathcal{C}$, the ∞-category $\mathcal{C}^{\mathrm{fil}}$ of filtered objects in $\mathcal{C}$ is defined as $\mathrm{Fun}(\mathbb{Z}^{\mathrm{fil}}, \mathcal{C})$. If $\mathcal{C}$ admits an symmetric monoidal structure $\mathcal{C}^{\otimes}$, we let $\left(\mathcal{C}^{\mathrm{fil}}\right)^{\otimes}$ denote $\mathcal{C}^{\mathrm{fil}}$ with symmetric monoidal structure given by Day convolution. [Lur17, Example 2.2.6.17] $\diamond$

The tensor product of two filtered objects $X, Y \in \mathcal{C}^{\mathrm{fil}}$ is given by

$$(X \otimes Y)_n \simeq \operatorname*{colim}_{i+j \to n} X_i \otimes Y_j.$$

Definition 3.3. Given a symmetric monoidal ∞-category $\mathcal{C}$, and an integer n, let $\mathcal{C}_n$ denote the full subcategory of $\mathcal{C}^{\mathrm{fil}}$ spanned by filtered objects $X \in \mathcal{C}^{\mathrm{fil}}$ satisfying:

1. $X_m \simeq 0$ for $m < 0$.

2. $X_0 \simeq 1_{\mathcal{C}}$.

3. $f_{m,m'} : X_m \to X_{m'}$ is an equivalence for $1 \leq m \leq m' \leq n$.

Furthermore let $\mathcal{C}_n^{\otimes}$ denote the ∞-operad spanned by objects $X_1 \oplus \cdots \oplus X_m$ with $X_i \in \mathcal{C}_n$.

$\diamond$

Remark 3.4. An object of $\mathcal{C}_n$ is equivalent to a filtered object, which on the first n terms are on the form

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & 1 \arrow[r, "u"] & X \arrow[r, "\mathrm{id}_X"] & X \arrow[r, "\mathrm{id}_X"] & \dots \arrow[r, "\mathrm{id}_X"] & X \\
 & {[-1]} & {[0]} & {[1]} & {[2]} & & {[n].}
\end{tikzcd}$$

$\mathcal{C}_n^{\otimes}$ is not an symmetric monoidal ∞-category, as $\mathcal{C}_n$ is not closed under tensor products.

$\diamond$

We will now show that $\mathbb{A}_{\mathrm{n}}$-algebra structures on $X \in \mathcal{C}_n$ get send to $\mathbb{A}_{\mathrm{n}}$-algebras structures on X_n. First we recall the definition of an $\mathbb{A}_{\mathrm{n}}$-operad map.

Definition 3.5. Given planar operads $\mathcal{C}^{\circledast}, \mathcal{D}^{\circledast}$ an $\mathbb{A}_n$-operad map, is a functor

$$f : \mathcal{C}^{\circledast} \times_{\mathrm{N}(\Delta^{\mathrm{op}}_{\leq n})} \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{D}^{\circledast} \times_{\mathrm{N}(\Delta^{\mathrm{op}}_{\leq n})} \mathrm{N}(\Delta^{\mathrm{op}})$$

with the following properties:

1. The diagram

$$\begin{tikzcd}
\mathcal{C}^{\circledast} \times_{\mathrm{N}(\Delta^{\mathrm{op}}_{\leq n})} \mathrm{N}(\Delta^{\mathrm{op}}) \arrow[rr, "f"] \arrow[dr] & & \mathcal{D}^{\circledast} \times_{\mathrm{N}(\Delta^{\mathrm{op}}_{\leq n})} \mathrm{N}(\Delta^{\mathrm{op}}) \arrow[dl] \\
 & \mathrm{N}(\Delta^{\mathrm{op}}_{\leq n}) &
\end{tikzcd}$$

   commutes.

2. For every inert morphism $\alpha : [m'] \to [m]$ in Δ satisfying $1 \leq m' \leq m \leq n$, the induced map $A(\alpha) : A([m]) \to A([m'])$ is a q-coCartesian morphism of $\mathcal{C}^{\circledast}$.

$\diamond$

Remark 3.6. An $\mathbb{A}_{\mathrm{n}}$-operad map f induces a functor

$$\mathrm{Alg}_{\mathbb{A}_{\mathrm{n}}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{A}_{\mathrm{n}}}(\mathcal{D})$$

by composition with f for every natural number n. $\diamond$

Lemma 3.7. *The composition*

$$\mathcal{C}_n^{\otimes} \hookrightarrow \left(\mathcal{C}^{\mathrm{fil}}\right)^{\otimes} \xrightarrow{\mathrm{ev}_n} \mathcal{C}^{\otimes}$$

is an $\mathbb{A}_n$*-operad map.*

Proof. We will to show that ev_n commutes with up to n-fold tensor products. That is given $n' \leq n$ and $F_1, \dots, F_{n'} \in \mathcal{C}_n$ we have

$$\left(F_1 \otimes \cdots \otimes F_{n'}\right)(n) \simeq F_1(n) \otimes \cdots \otimes F_{n'}(n).$$

From the formula of Day convolution, the tensor product of n' elements evaluated at n is

$$(F_1 \otimes \cdots \otimes F_{n'})(n) = \operatorname*{colim}_{i_1 + \dots i_{n'} < n} F_1(i_1) \otimes \dots \otimes F_{n'}(i_{n'}).$$

We can in this case disregard all objects in the colimit where some $i_j > 1$, since all maps

$$F_1(i_1) \otimes \cdots \otimes F_j(1) \otimes \cdots \otimes F_{n'}(i_{n'}) \to F_1(i_1) \otimes \cdots \otimes F_j(i_j) \otimes \cdots \otimes F_{n'}(i_{n'})$$

are equivalences, and as such does not affect the colimit, so we have

$$(F_1 \otimes \cdots \otimes F_{n'})(n) \simeq \operatorname*{colim}_{(i_j) \in \{0,1\}} F_1(i_1) \otimes \cdots \otimes F_{n'}(i_{n'}).$$

The diagram the colimit is taken over, is the simplicial set $(\Delta^1)^{n'}$. In this case the map $\Delta^0 \subseteq (\Delta^1)^{n'}$ to the vertex $(1, \dots, 1)$ is cofinal, since the horn inclusion $\Lambda^1_1 \subseteq \Delta^1$ is right anodyne, and right anodyne maps are closed under compositions and products with simplicial sets. It follows that

$$(F_1 \otimes \cdots \otimes F_{n'})(n) \cong F_1(1) \otimes \cdots \otimes_{\mathcal{C}} F_{n'}(1) \cong F_1(n) \otimes \cdots \otimes_{\mathcal{C}} F_{n'}(n)$$

for all $1 \leq n' \leq n$, showing that ev_n is $\mathbb{A}_n$-monoidal on the subcategory $\mathcal{C}_n^{\otimes}$. □

$$\begin{tikzcd}
F_1(0) \otimes F_2(0) \arrow[r] \arrow[d] & F_1(1) \otimes F_2(0) \arrow[r, "\simeq"] \arrow[d] & F_1(2) \otimes F_2(0) \\
F_1(0) \otimes F_2(1) \arrow[r] \arrow[d, "\simeq"'] & F_1(1) \otimes F_2(1) & \\
F_1(0) \otimes F_2(2) & &
\end{tikzcd}$$

Figure 2: The colimit diagram for $(F_1 \otimes F_2)(1)$, with $F_1(1) \otimes F_2(1)$ being final.

3.2 Obstruction theory

In [Bur22, Proposition 2.4], an inductively defined obstruction theory $\{\theta_k\}_{k\geq 2}$ is constructed for a map $r : X \to 1_{\mathcal{C}}$ in a stable symmetric monoidal ∞-category, such that the vanishing of all obstructions $\{\theta_k\}_{k\geq 2}$ give an $\mathbb{A}_\infty$-algebra structure on $1_{\mathcal{C}}/r$.

Proposition 3.8 ([Bur22], $\mathbb{A}_\infty$-version)**.** *Given a map $r : X \to 1_{\mathcal{C}}$ in $\mathcal{C}$, there exists a sequence of inductively defined obstructions*

$$\theta_k \in [\Sigma^{-3}(\Sigma^2 X), 1_{\mathcal{C}}/r] \qquad \textit{for} \quad k \geq 2,$$

whose vanishing allows us to inductively construct a sequence of $\mathbb{A}_\infty$-algebras

$$1_{\mathcal{C}} = \overline{R}^0 \xrightarrow{\overline{r}_1} \overline{R}^1 \xrightarrow{\overline{r}_1} \overline{R}^2 \to \cdots \to 1_{\mathcal{C}}/r$$

converging to an $\mathbb{A}_\infty$-algebra structure on $1_{\mathcal{C}}/r$, where each map $\overline{r}_k$ sits in a pushout square

$$\begin{tikzcd}
1_{\mathcal{C}}\{\Sigma^{-2}(\Sigma^2 X)\} \arrow[r, "\mathrm{aug}"] \arrow[d, "\overline{s}_k"] & 1_{\mathcal{C}} \arrow[d] \\
\overline{R}^{k-1} \arrow[r, "\overline{r}_k"] & \overline{R}^k \arrow[ul, phantom, "\lrcorner", very near start].
\end{tikzcd}$$

In the proof of this proposition, it is further shown that if the obstructions $\theta_2, \ldots, \theta_n$ vanish, we get an $\mathbb{A}_\infty$ structure on a filtered object $\widetilde{R}^n \in \mathcal{C}_n$ with $(\widetilde{R}^n)_n \simeq 1_{\mathcal{C}}/r$. The image of $\widetilde{R}^n$ under the functor

$$\mathrm{Alg}_{\mathbb{A}_\infty}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{A}_\mathrm{n}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{A}_\mathrm{n}}(\mathcal{D})$$

is then an $\mathbb{A}_\mathrm{n}$-algebra with $1_{\mathcal{C}}/r$ as the underlying object. It follows that the vanishing of the obstructions $\theta_2, \ldots, \theta_n$, implies the existence of an $\mathbb{A}_n$-ring structure on $1_{\mathcal{C}}/r$. Summing up, we get the following proposition.

Proposition 3.9. *Given a map $r : X \to 1_{\mathcal{C}}$ in $\mathcal{C}$, there exists a sequence of inductively defined obstructions*

$$\theta_k \in [\Sigma^{2k-3} X^{\otimes k}, 1_{\mathcal{C}}/r],$$

such that the vanishing of $\theta_1, \ldots, \theta_n$ induces an $\mathbb{A}_n$-algebra structure on $1_{\mathcal{C}}/r$ with unit given by the cofiber map.

3.3 Alternate construction

We here sketch another construction of the obstruction theory based on an unital version of the following theorem by Lurie in [Lur17, Theorem 4.1.6.8].

Theorem 3.10. *Let $\mathcal{C}$ be a monoidal ∞-category, let A be an object of $\mathcal{C}$, and let $n \geq 2$. Then there is a pullback diagram of ∞-categories*

$$\begin{tikzcd}
\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_n}(\mathcal{C}) \times_{\mathcal{C}} \{A\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{K_n} \arrow[d] \\
\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_{n-1}}(\mathcal{C}) \times_{\mathcal{C}} \{A\} \arrow[r, "\beta"] & \mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{\partial K_n}.
\end{tikzcd}$$

Where $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_n}(\mathcal{C})$ is the ∞-category of non-unital $\mathbb{A}_n$-algebras in $\mathcal{C}$.

This theorem states informally, that the additional data to extend a nonunital $\mathbb{A}_{n-1}$-algebra to a nonunital $\mathbb{A}_{\mathrm{n}}$-algebra, is a multiplication map of n elements $\mu_n : A^{\otimes n} \to A$, such that it is compatible with the $\mathbb{A}_{n-1}$-structure.

A similar result should hold for unital algebras, but in this case the extension should also respect the unital structure, so that there are homotopies $\mu_n \circ i_m \sim \mu_{n-1}$, where $i_m : A^{\otimes n-1} \to A^{\otimes n}$ is the map giving by the unit map in the m'th coordinate. The following construction allows us to encode these homotopies.

Construction 3.11 ([Bha22])**.** Let $[n]$ denote the category of ordered subsets of $\{1, \dots, n\}$, and $C(n, n-1)$ be the full subcategory consisting of subsets of at most order $n-1$. Given an object A in a monoidal ∞-category $\mathcal{C}$, we have a functor

$$F^A_{n,n-1} : C(n, n-1) \to \mathcal{C},$$

sending a set of order i to $A^{\otimes i}$, and sending an inclusion $i \subseteq j$ to the map $u_{I,J} : A^{\otimes i} \to A^{\otimes j}$ given by id_A for $j \in I$ and the unit map for $j \notin I$. Let T be the colimit of $F^A_{n,n-1}$. $\diamond$

By the inclusions of the unit, we get a map $T \to A^{\otimes n}$. A unital version of [Lur17, Theorem 4.1.6.8] should then be the following.

Claim 3.12. *Let $\mathcal{C}$ be a monoidal ∞-category, let A be an object of $\mathcal{C}$, let $\varphi : 1_{\mathcal{C}} \to A$ be a map from the unit, and let $n \geq 2$. Then there is a natural pullback diagram of ∞-categories*

$$\begin{tikzcd}
\mathrm{Alg}_{\mathbb{A}_n}(\mathcal{C}) \times_{\mathcal{C}_{1/}} \{1 \xrightarrow{u} A\} \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{K_n} \arrow[d] \\
\mathrm{Alg}_{\mathbb{A}_{n-1}}(\mathcal{C}) \times_{\mathcal{C}_{1/}} \{1 \xrightarrow{u} A\} \arrow[r, "\beta"] & \mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{\partial K_n} \times_{\mathrm{Map}_{\mathcal{C}}(T,A)^{\partial K_n}} \mathrm{Map}_{\mathcal{C}}(T, A)^{K_n}.
\end{tikzcd}$$

This result seems harder to prove than Theorem 3.10, as the subcategory $\Delta^s_{\leq n} \hookrightarrow \Delta_{\leq n}$ only having the injective morphisms in $\Delta_{\leq n}$, which is used to define $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{A}_{\mathrm{n}}}(\mathcal{C})$, is much simpler than $\Delta_{\leq n}$, as the morphisms only go in one direction. We will show how this claim leads to the obstruction theory when $\mathcal{C}$ is a presentable stable ∞-category.

Construction 3.13. Let $\mathcal{C}$ be a presentable stable monoidal ∞-category, and $u : 1_{\mathcal{C}} \to A$ be a map from the unit to an object. Since Sp is the unit in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, every stable ∞-category is left-tensored over Sp, and further enriched over Sp by [Lur17, Proposition 4.2.1.33]. In this case we have an equivalence

$$\mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{K_n} \cong \mathrm{Map}_{\mathcal{C}}(\Sigma^\infty K_n \otimes A^{\otimes n}, A),$$

and similar for the other mapping spectra

$$\begin{gathered}\mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{\partial K_n} \times_{\mathrm{Map}_{\mathcal{C}}(T,A)^{\partial K_n}} \mathrm{Map}_{\mathcal{C}}(T, A)^{K_n} \\ \cong \mathrm{Map}_{\mathcal{C}}\left(\Sigma^\infty \partial K_n \otimes A^{\otimes n} \amalg_{\Sigma^\infty \partial K_n \otimes T} \Sigma^\infty K_n \otimes T, A\right).\end{gathered}$$

We can then take the fiber of the map

$$\Sigma^\infty \partial K_n \otimes A^{\otimes n} \amalg_{\Sigma^\infty \partial K_n \otimes T} \Sigma^\infty K_n \otimes T \to \Sigma^\infty K_n \otimes A^{\otimes n},$$

which is equivalent to the desuspension of the cofiber. This is then the total cofiber of the diagram.

$$\begin{tikzcd} \Sigma^\infty \partial K_n \otimes T \arrow[r] \arrow[d] & \Sigma^\infty K_n \otimes T \arrow[d] \\ \Sigma^\infty \partial K_n \otimes A^{\otimes n} \arrow[r] & \Sigma^\infty K_n \otimes A^{\otimes n}, \end{tikzcd}$$

which is

$$\Sigma^\infty (K_n/\partial K_n) \otimes (A^{\otimes n}/T) \cong \Sigma^{n-2}(A/u)^{\otimes n}.$$

From this we get a cofiber/fiber sequence

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{K_n} \to \mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{\partial K_n} &\times_{\mathrm{Map}_{\mathcal{C}}(T,A)^{\partial K_n}} \mathrm{Map}_{\mathcal{C}}(T, A)^{K_n} \\ &\xrightarrow{q} \mathrm{Map}_{\mathcal{C}}(\Sigma^{n-3}(A/u)^{\otimes n}, A).\end{aligned}$$

◇

Remark 3.14. Given an $\mathbb{A}_{n-1}$-algebra A in $\mathcal{C}$ with unit $u : 1_{\mathcal{C}} \to A$, we have a diagram

$$\begin{tikzcd} & \mathrm{Alg}_{\mathbb{A}_n}(\mathcal{C}) \times_{\mathcal{C}_{1/}} \{1 \xrightarrow{u} A\} \arrow[r] \arrow[d] & \mathrm{Map}(A^{\otimes n}, A)^{K_n} \arrow[d] \\ * \arrow[r, "A"] & \mathrm{Alg}_{\mathbb{A}_{n-1}}(\mathcal{C}) \times_{\mathcal{C}_{1/}} \{1 \xrightarrow{u} A\} \arrow[r, "\beta"] & \mathrm{Map}(A^{\otimes n}, A)^{\partial K_n} \times_{\mathrm{Map}(T,A)^{\partial K_n}} \mathrm{Map}(T, A)^{K_n}. \end{tikzcd}$$

From Claim 3.12, giving an extension of A to an $\mathbb{A}_{\mathrm{n}}$-algebra, is equivalent to a lift of $\beta(A)$ to $\mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{K_n}$. Since $\mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{K_n}$ is the fiber of the map

$$\mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{\partial K_n} \times_{\mathrm{Map}_{\mathcal{C}}(T,A)^{\partial K_n}} \mathrm{Map}_{\mathcal{C}}(T, A)^{K_n} \xrightarrow{q} \mathrm{Map}_{\mathcal{C}}(\Sigma^{n-3}(A/u)^{\otimes n}, A),$$

the space of extensions of A to an $\mathbb{A}_{\mathrm{n}}$-algebra structure is equivalent to space of nulhomotopies of $q\beta(A)$. So $q\beta(A)$ is an obstruction for extending $\mathbb{A}_{n-1}$-algebra structures.

If A is the cofiber of a map to the unit $X \to 1_{\mathcal{C}}$, then the obstruction lies in the same group as the obstruction in Proposition 5.1.

$\diamond$

4 Synthetic Spectra

We will apply the obstruction theory from Proposition 3.9, to show that $\mathbb{S}/4$ admits an $\mathbb{A}_5$-algebra structure. Since $\mathbb{S}/4$ already admits an $\mathbb{A}_4$-algebra structure, the relevant obstruction is $\theta_5 \in \pi_7(\mathbb{S}/4) \neq 0$. The obstructions are hard to calculate concretely, and since θ_5 does not lie in a null-group, it does not vanish automatically. We will instead apply Proposition 3.9 to a stable ∞-category related to Sp.

To do this we introduce the ∞-category of synthetic spectra Syn_E with regards to an Adams spectrum E introduced in [Pst22]. We follow the exposition given in [BHS19], where the focus is not on the construction of Syn_E, but on what properties it satisfies. We refer the reader to both of these sources for a more detailed account of synthetic spectra.[3]

Definition 4.1. Suppose that E is a homotopy associative ring spectrum such that E_* and E_*E are graded commutative rings. Following [Pst22, Definition 3.12], we say that a finite spectrum X is *finite E_*-projective* (or simply *finite projective* if E is clear from context) if E_*X is a projective E_*-module. We say that E is of *Adams type* if E can be written as a filtered colimit of finite projective spectra E_α such that the natural maps

$$E^*E_\alpha \to \mathrm{Hom}_{E_*}(E_*E_\alpha, E_*)$$

are isomorphisms. $\diamond$

Construction 4.2 (Pstrągowski)**.** Let E denote an Adams type homology theory. Then there is a stable, presentable symmetric monoidal ∞-category Syn_E together with a functor

$$\nu_E : \mathrm{Sp} \to \mathrm{Syn}_E,$$

which is lax symmetric monoidal and preserves filtered colimits [Pst22, Lemma 4.4]. However, ν_E does *not* preserve cofiber sequences in general. When E is clear from context, we will often denote ν_E by ν. $\diamond$

Example 4.3. The spectrum $\mathrm{H}\mathbb{F}_2$ is of Adams type, which is the only example we will use. In this case the functor $\upsilon_{\mathrm{H}\mathbb{F}_2}$ is symmetric monoidal [BHS19, Remark 9.5]. $\diamond$

Since ν does not preserve all cofiber sequences, the synthetic spectra $\Sigma\nu(\mathbb{S})$ and $\nu(\Sigma\mathbb{S})$ are not isomorphic. We therefore have two different gradings on Syn_E.

[3] Note our convention of bigraded spheres matches [Bur22] and not the above sources. In particular $\mathbb{S}^{a,b}$ correspond to $\mathbb{S}^{a,a+b}$ in [BHS19] and [Pst22].

Definition 4.4. The bigraded sphere $\mathbb{S}^{a,b}$ is defined to be $\Sigma^{-b}\nu(\mathbb{S}^{a+b})$. The map from the universal property of the pushout

$$\mathbb{S}^{0,-1} = \Sigma\nu(\mathbb{S}^{-1}) \to \nu(\Sigma\mathbb{S}^{-1}) = \mathbb{S}^{0,0}$$

is denoted by τ. The cofiber of τ is denoted $C\tau$. A synthetic spectrum X is *τ-invertible* if the map $\mathrm{id}_X \otimes \tau : \Sigma^{0,1}X \to X$ is invertible. The ∞-category $\mathrm{Syn}_E(\tau^{-1})$ is the full subcategory of τ-invertible synthetic spectra. $\diamond$

The following theorem summarises the properties of the ∞-category of synthetic spectra we will use in this thesis.

Theorem 4.5 (Pstrągowski)**.**

1. *The localization functor given by inverting τ is symmetric monoidal.*
2. *The full subcategory of τ-invertible synthetic spectra is equivalent to the category of spectra.*
3. *The composite $\tau^{-1} \circ \nu$ is equivalent to the identity functor on* Sp.
4. *The object $C\tau$ admits the structure of an $\mathbb{E}_\infty$-ring in* Syn_E.
5. *Suppose that E is homotopy commutative. Then there is a natural fully faithful, monoidal functor*
$$\mathrm{Mod}_{C\tau} \to \mathrm{Stable}_{E_*E},$$
where the target is Hovey's stable ∞-category of comodules and the composition of $\nu(-) \otimes C\tau$ with this functor is equivalent to $E_*(-)$.

We can construct the following diagram, where every arrow except ν and $E_*(-)$ is a left adjoint.

$$\begin{tikzcd}
 & \mathrm{Sp} \arrow[dl, "\mathrm{id}_{\mathrm{Sp}}"'] \arrow[d, "\nu"] \arrow[drr, bend left, "E_*(-)"] & & \\
\mathrm{Sp} & \mathrm{Syn}_E \arrow[l, "\tau^{-1}"] \arrow[r, "-\otimes C\tau"] & \mathrm{Mod}_{C\tau} \arrow[r] & \mathrm{Stable}_{E_*E}
\end{tikzcd}$$

The reason we introduce synthetic spectra, is that it allows us to keep track of the Adams filtrations of maps.

Definition 4.6 (Adams Filtration)**.** Let $f : X \to Y$ be a map between spectra, and E be a spectrum. The map f has Adams filtration $\geq n$ with regards to E, if f can be written as a composite of maps

$$X = X_0 \to ... \to X_n = Y,$$

where each map in the composite induces the 0 map on homology with E. $\diamond$

Remark 4.7. The filtration matches the filtration from the Adams spectral sequence for E. ◇

Lemma 4.8 ([BHS19] Lemma 9.15)**.** *If a map $f : X \longrightarrow Y$ of spectra has E-Adams filtration k, then there exists a factorization in E-synthetic spectra*

$$\begin{tikzcd} & \nu(Y) \arrow[d, "\tau^k"] \\ \Sigma^{0,k}\nu(X) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "\nu(f)"] & \Sigma^{0,k}\nu(Y). \end{tikzcd}$$

Example 4.9. The map $2 : \mathbb{S} \to \mathbb{S}$ has Adams filtration 1, so there is a map $\tilde{2} : \mathbb{S}^{0,1} \to \nu\mathbb{S}$ with $\tau\tilde{2} = 2$. Note that $\tau^{-1}(\mathbb{S}^{0,0}/\tilde{2}^n) \simeq \mathbb{S}/2^n$. Since localization by τ is a symmetric monoidal functor, an $\mathbb{A}_n$-structure on $\nu\mathbb{S}/\tilde{4}$ induces a $\mathbb{A}_n$-structure on $\nu\mathbb{S}/4$. ◇

Proposition 4.10. *The synthetic spectrum $\nu\mathbb{S}/4$ admits an $\mathbb{A}_2$-multiplication.*

Proof. Applying the obstruction theory from Proposition 3.9 to the map $\tilde{4} : \mathbb{S}^{0,2} \to \nu\mathbb{S}$ in $\mathrm{Syn}_{\mathbb{F}_2}$, we get the obstructions

$$\theta_k \in [\nu\mathbb{S}^{2k-3,3}, \nu\mathbb{S}/\tilde{4}] = \pi_{2k-3,3}(\nu\mathbb{S}/\tilde{4})$$

for $k \geq 2$.[4] From the calculation of the E_2-page of the Adams spectral sequence in [IWX22], we get that there is no differentials in topological degree less than or equal to 12. [BHS19, Theorem A.8] then implies that in this range the homotopy groups $\pi_{t,s}(\nu\mathbb{S})$ are given by the E_2-page of the $\mathbb{F}_2$-Adams spectral sequence of $\mathbb{S}$ tensored with $\mathbb{Z}[\tau]$. From this, we can calculate the bigraded homotopy groups of $\nu\mathbb{S}/\tilde{4}$ pictured below.

[4]Each obstruction is only defined by a nulhomotopy of the previous obstruction, so they are not uniquely defined, and only exists if the previous obstruction vanishes.

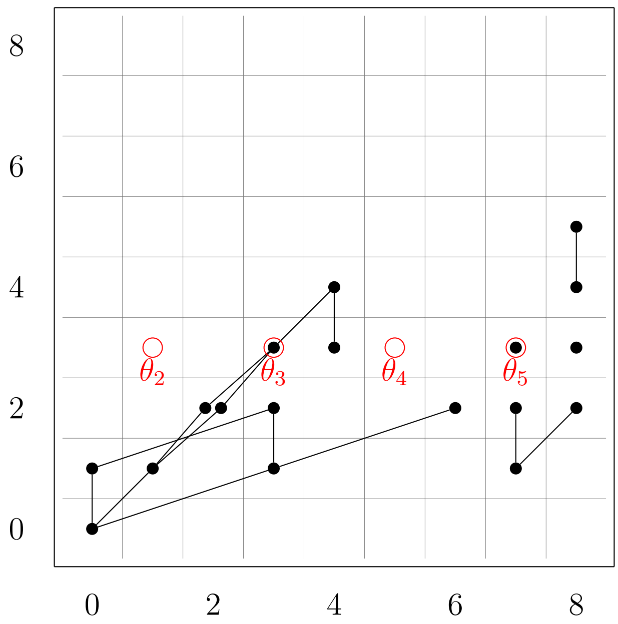


Figure 3: Bigraded homotopy groups of $\nu\mathbb{S}/\tilde{4}$ without τ-multiples. Each dot represents a copy of $\mathbb{F}_2$

Since $\tau \in \pi_{0,-1}(\nu\mathbb{S})$, any τ-multiple lives below non τ-multiples. We then see from the above diagram, that the obstruction θ_2 vanishes since it lives in a null-group. $\square$

The location of an eventual third and fifth obstruction does not lie in a null-group, so they do not vanish formally. The next sections are dedicated to showing these obstructions vanish.

5 Relating Obstructions in different categories

In Section 4, we saw the obstructions θ_3, θ_5 for $\nu\mathbb{S}/\tilde{4}$ being a $\mathbb{A}_3$- and $\mathbb{A}_5$-ring spectrum respectively, lies in the bigraded homotopy groups $\pi_{3,3}, \pi_{7,3}$ for $(\nu\mathbb{S}/\tilde{4})$ which are both non-trivial.

To remedy the situation, we will show that these obstructions factors through the quotient map $\nu\mathbb{S} \to \nu\mathbb{S}/\tilde{2}^2$, which is null on both $\pi_{3,3}, \pi_{7,3}$, implying the obstructions vanishes. To do so, we will establish a functoriality of the obstruction theory of Proposition 3.9 and use it on another stable ∞-category.

5.1 Map between Obstructions

A monoidal functor F induces a functor on ∞-categories of $\mathbb{A}_n$-algebras. The following proposition shows that F also maps the obstruction from Proposition 3.9 to the obstruction on the target.

Proposition 5.1. *Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be an exact monoidal functor of stable monoidal ∞-categories, and let $A \in \mathrm{Alg}_{\mathbb{A}_{k-1}}(\mathcal{C})$ be an $\mathbb{A}_{k-1}$-algebra with unit $u : 1_{\mathcal{C}} \to A$.*

The map F sends the obstruction θ_k from Proposition 3.9 for A to the obstruction for extending the induced $\mathbb{A}_{k-1}$-algebra structure on $F(A)$ to a $\mathbb{A}_k$-structure by the map

$$\mathrm{Map}_{\mathcal{C}}(\Sigma^{2k-3}(A/u^{\otimes k}, A) \xrightarrow{F} \mathrm{Map}_{\mathcal{D}}(\Sigma^{2k-3}F(A)/F(u)^{\otimes k}, F(A)).$$

If F admits a right adjoint R, the above map factorises as

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(\Sigma^{2k-3}A/u^{\otimes k}, A) &\xrightarrow{\eta_A \circ -} \mathrm{Map}_{\mathcal{C}}(\Sigma^{2k-3}A/u^{\otimes k}, RF(A)) \\ &\xrightarrow{\Psi} \mathrm{Map}_{\mathcal{D}}(\Sigma^{2k-3}F(A)/F(u)^{\otimes k}, F(A)).\end{aligned}$$

Where Ψ is adjunction map, and η is the unit of the adjunction.

Proof. From the naturality of Claim 3.12, The obstruction of A is mapped to the obstruction of $F(A)$ by the map given by the universal property

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{K_n} \arrow[r, "F"] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(F(A)^{\otimes n}, F(A))^{\partial K_n} \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(A^{\otimes n}, A)^{\partial K_n} \times_{\mathrm{Map}_{\mathcal{C}}(T,A)^{\partial K_n}} \mathrm{Map}_{\mathcal{C}}(T, A)^{K_n} \arrow[r, "F"] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(F(A)^{\otimes n}, F(A))^{\partial K_n} \times_{\mathrm{Map}_{\mathcal{C}}(T,F(A))^{\partial K_n}} \mathrm{Map}_{\mathcal{C}}(T, F(A))^{K_n} \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}\left(\Sigma^{2k-3}A/u^{\otimes k}, A\right) \arrow[r, dashed] & \mathrm{Map}_{\mathcal{C}}(\Sigma^{2k-3}F(A)/F(u)^{\otimes k}, A).
\end{tikzcd}$$

Since the two first horizontal maps are given by F, the induced map is also given by F. The statement of right adjoints, follows from the definition. $\square$

Example 5.2. Let A, B be $\mathbb{E}_2$-algebras in a symmetric monoidal category $\mathcal{C}^{\otimes}$, and let $\varphi : A \to B$ be a morphism in $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C})$. Then the functor

$$\begin{aligned}\mathrm{RMod}_A(\mathcal{C}) &\to \mathrm{RMod}_B(\mathcal{C}) \\ M &\mapsto M \otimes_A B\end{aligned}$$

is $\mathbb{E}_1$-monoidal by [Lur17, Theorem 4.8.5.16], with right adjoint given by restriction of scalars. $\diamond$

5.2 $\mathbb{E}_\infty$-Rings from the Thom construction

We will now construct a map of $\mathbb{E}_\infty$-rings to use Proposition 5.1 and Example 5.2. To do so we use the Thom construction given in [Car+23]. While [Car+23] only deals with groups, the construction only uses the underlying monoid structure, and so works equally well for monoids.

Definition 5.3. Given a presentably symmetric monoidal ∞-category $\mathcal{C}$, let $\mathrm{Pic}(\mathcal{C})$ denote the subgroupoid of $\mathcal{C}$, spanned by invertible objects and equivalences between them with $\mathbb{E}_\infty$-group structure given by the tensor product. The Thom functor

$$\begin{aligned} \mathrm{Th}_{\mathcal{C}} : \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{S})_{/\,\mathrm{Pic}(\mathcal{C})} &\to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \\ (\varphi : X \to \mathrm{Pic}(\mathcal{C})) &\to \mathrm{colim}_X\,(i \circ \varphi) \end{aligned}$$

is constructed in [Car+23], where $i : \mathrm{Pic}(\mathcal{C}) \hookrightarrow \mathcal{C}$ denotes the inclusion. $\diamond$

The following proposition gives a unique characterization of $\mathrm{Th}_{\mathcal{C}}$ up to isomorphism.

Proposition 5.4 ([Car+23]). *Let M be an $\mathbb{E}_n$-monoid and suppose that for every $\mathcal{C} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, we have a functor*

$$\mathrm{Th}'_{\mathcal{C}} : \mathrm{Map}_{\mathbb{E}_n}(M, \mathrm{Pic}(\mathcal{C})) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$$

lifting colim_M along $\mathrm{Alg}_n(\mathcal{C}) \to \mathcal{C}$, and such that for every $F : \mathcal{C} \to \mathcal{D} \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, we have a natural isomorphism

$$F(\mathrm{Th}_{\mathcal{C}}(\xi)) \simeq Th_{\mathcal{D}}(F(\xi)).$$

Then for every $\mathcal{C}$,, we have an isomorphism of functors $\mathrm{Th}'_{\mathcal{C}} \simeq \mathrm{Th}_{\mathcal{C}}$.

Notation 5.5. *We will write $\mathrm{Th}_{\mathcal{C}}$ as Th, when $\mathcal{C}$ is obvious.*

Corollary 5.6. *The compositions*

$$\mathrm{Map}_{\mathbb{E}_\infty}(\mathbb{N}, \mathrm{Pic}(\mathcal{C})) \xrightarrow{\mathrm{Th}_{\mathcal{C}}} \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \xrightarrow{\mathrm{res}} \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$$

and

$$\mathrm{Map}_{\mathbb{E}_\infty}(\mathbb{N}, \mathrm{Pic}(\mathcal{C})) \xrightarrow{\mathrm{ev}_{\langle 1\rangle}} \mathcal{C} \xrightarrow{\mathrm{Free}_{\mathbb{E}_1}} \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$$

are isomorphic.

Proof. The functor $\mathrm{Free}_{\mathbb{E}_1}$ exists and agree with $\mathrm{colim}_{\mathbb{N}}$ in $\mathcal{C}$ by [Lur17, Proposition 4.1.1.18], so both compositions are lift of $\mathrm{colim}\,\mathbb{N}$. Furthermore given $F \in \mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ with right adjoint R, we have the diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \arrow[d, "\theta"'] & \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{D}) \arrow[l, "R"] \arrow[d, "\theta"] \\ \mathcal{C} & \mathcal{D} \arrow[l, "R"] \end{tikzcd}$$

so $\mathrm{Free}_{\mathbb{E}_1} \circ F \simeq F \circ \mathrm{Free}_{\mathbb{E}_1}$, since they are left adjoints to isomorphic functors. It follows that both $\mathrm{res} \circ \mathrm{Th}_{\mathcal{C}}$ and $\mathrm{Free}_{\mathbb{E}_1} \circ \mathrm{ev}_{\langle 1\rangle}$ uphold the properties of Proposition 5.4 and are therefore isomorphic. $\square$

We will now calculate the Picard groups for certain symmetric monoidal ∞-categories.

Lemma 5.7. *Given functors in* $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}$$

with F fully faithful and $GF \simeq \mathrm{id}_{\mathcal{C}}$, we get a splitting in $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$.

$$\mathrm{Pic}(\mathcal{D}) \simeq \mathrm{Pic}(\mathcal{C}) \oplus G$$

where G is discrete.

Proof. Since F, G are symmetric monoidal, we get a splitting

$$\mathrm{Pic}(\mathcal{C}) \underset{\mathrm{Pic}(G)}{\overset{\mathrm{Pic}(F)}{\rightleftarrows}} \mathrm{Pic}(\mathcal{D}) \longrightarrow \mathrm{cofib}(\mathrm{Pic}(F)).$$

We then have an equivalence

$$\mathrm{Pic}(D) \simeq \mathrm{Pic}(\mathcal{C}) \oplus \mathrm{cofib}(\mathrm{Pic}(F)).$$

Since F is fully faithful, $\pi_n \mathrm{Pic}(F)$ is an isomorphism for $n \neq 0$, and $\pi_0 \mathrm{Pic}(F)$ is an injection, so $\pi_n \mathrm{cofib}(\mathrm{Pic}(F))$ vanishes for $n \neq 0$. $\square$

Example 5.8. For $\mathrm{Syn}_{\mathbb{F}_2}$, we have the splitting

$$\mathrm{Sp} \xrightarrow{\nu} \mathrm{Syn}_{\mathbb{F}_2} \xrightarrow{\tau^{-1}} \mathrm{Sp}$$

with $\tau^{-1}\nu \simeq 1$ and ν fully faithful, so we have an equivalence

$$\mathrm{Pic}(\mathrm{Syn}_{\mathbb{F}_2}) \simeq \mathrm{Pic}(\mathrm{Sp}) \oplus G_{\mathrm{Syn}_{\mathbb{F}_2}}$$

with $G_{\mathrm{Syn}_{\mathbb{F}_2}}$ discrete. The synthetic spectrum $\mathbb{S}^{0,2}$ is invertible, with inverse $\mathbb{S}^{0,-2}$, so $\mathbb{S}^{0,-2} \in \mathrm{Pic}(\mathrm{Syn}_{\mathbb{F}_2})$. Since $\tau^{-1}(\mathbb{S}^{0,-2}) \simeq \mathbb{S}$, we have $\mathbb{S}^{0,2}$ is send to the sphere $\mathbb{S}$ in Sp, so we have $\mathbb{S}^{0,2} \in G_{\mathrm{Syn}_{\mathbb{F}_2}}$. $\diamond$

Definition 5.9. Let $\mathbb{Z}^{\mathrm{ds}}$ be the symmetric monoidal category with objects being the integers, and morphisms being only identity morphisms, where the symmetric monoidal structure is given by addition on the integers.

For an ∞-category $\mathcal{C}$, the ∞-category of graded objects $\mathcal{C}^{\mathrm{Gr}}$ is defined as the functor category $\mathrm{Fun}(\mathbb{Z}^{\mathrm{ds}}, \mathcal{C})$. If $\mathcal{C}$ is (symmetric) monoidal, then $\mathcal{C}^{\mathrm{Gr}}$ is given the (symmetric) monoidal structure from Day convolution. $\diamond$

Example 5.10. For $\mathcal{C}^{\mathrm{Gr}}$, we have the splitting

$$\mathcal{C} \xrightarrow{i_0} \mathcal{C}^{\mathrm{Gr}} \xrightarrow{\mathrm{colim}} \mathcal{C},$$

where i_0 is the inclusion in the 0'th coordinate. We then have an equivalence

$$\mathrm{Pic}(\mathcal{C}^{\mathrm{Gr}}) \simeq \mathrm{Pic}(\mathcal{C}) \oplus G_{\mathcal{C}^{\mathrm{Gr}}}$$

with $G_{\mathcal{C}^{\mathrm{Gr}}}$ discrete. The graded object $1_{\mathcal{C}}(1)$ is invertible in $\mathcal{C}^{\mathrm{Gr}}$, with inverse $1_{\mathcal{C}}(-1)$, so $1_{\mathcal{C}}(1) \in \mathrm{Pic}(\mathrm{Syn}_{\mathbb{F}_2})$. Since $\mathrm{colim}\,(1_{\mathcal{C}}(1))\,, \simeq 1_{\mathcal{C}}$ The graded object $1_{\mathcal{C}}(1)$ maps to the unit in $\mathcal{C}$, so we have $1_{\mathcal{C}}(1) \in G_{\mathcal{C}^{\mathrm{Gr}}}$. ◇

Example 5.11. Combining the two above examples, we get an equivalence

$$\mathrm{Pic}(\mathrm{Syn}_{\mathbb{F}_2}^{\mathrm{Gr}}) \simeq \mathrm{Pic}(\mathrm{Sp}) \oplus G'$$

with $\mathbb{S}^{0,2}(1) \in G' = G_{\mathrm{Syn}_{\mathbb{F}_2}^{\mathrm{Gr}}} \oplus G_{\mathrm{Syn}_{\mathbb{F}_2}}$. Since G' is discrete, we get a map of monoids $\mathbb{N} \to G' \to \mathrm{Pic}(\mathrm{Syn}_{\mathbb{F}_2}^{\mathrm{Gr}})$, sending 1 to $\mathbb{S}^{0,2}(1)$. ◇

Construction 5.12. We have an adjunction

$$\mathrm{Sp}_{\geq 0}^{\mathrm{Gr}} \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} \mathrm{Sp}^{\mathrm{Gr}}$$

with i_0 symmetric monoidal, so $\tau_{\geq 0}$ is lax monoidal, giving a functor

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}^{\mathrm{Gr}}) \xrightarrow{\tau_{\geq 0}} \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}_{\geq 0}^{\mathrm{Gr}}).$$

◇

Definition 5.13. We let R^{Gr} denote the $\mathbb{E}_\infty$-algebra given by the image of the map $\varphi : \mathbb{N} \to \mathrm{Pic}(\mathrm{Syn}_{\mathbb{F}_2}^{\mathrm{Gr}})$ sending 1 to $\mathbb{S}^{0,2}(1)$ by the functor

$$\mathrm{Map}_{\mathbb{E}_\infty}(\mathbb{N}, \mathrm{Pic}(\mathrm{Syn}_{\mathbb{F}_2}^{\mathrm{Gr}})) \xrightarrow{\mathrm{Th}} \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Syn}_{\mathbb{F}_2}^{\mathrm{Gr}}) \xrightarrow{\tau_{\geq 0}} \mathrm{Alg}_{\mathbb{E}_\infty}((\mathrm{Syn}_{\mathbb{F}_2}^{\mathrm{Gr}})_{\geq 0}),$$

and R be given by the image of R^{Gr} under the map

$$\mathrm{Alg}_{\mathbb{E}_\infty}((\mathrm{Syn}_{\mathbb{F}_2}^{\mathrm{Gr}})_{\geq 0}) \xrightarrow{\mathrm{colim}} \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Syn}_{\mathbb{F}_2}).$$

◇

Remark 5.14. Since the Thom functor on underlying objects is given by $\mathrm{colim}_{\mathbb{N}}$, the underlying graded synthetic spectrum of R^{Gr} is $\bigoplus_{n\in\mathbb{N}} \mathbb{S}^{0,2n}(n)$. Furthermore R must have underlying synthetic spectrum given by $\bigoplus_{n\in\mathbb{N}_0} \mathbb{S}^{0,2n}$, with homotopy groups $\pi_{*,*}R = \pi_{*,*}\nu\mathbb{S}[x]$ with x sitting in bidegree $(0,2)$. ◇

Remark 5.15. We have an adjunction

$$(\mathrm{Syn}_{\mathbb{F}_2}^{\mathrm{Gr}})_{\geq 0} \underset{i_0}{\overset{p_0}{\rightleftarrows}} \mathrm{Syn}_{\mathbb{F}_2},$$

where both i_0 and p_0 are monoidal, so the unit and counit of the adjunction is monoidal. Applying the unit to R^{Gr}, we then get a $\mathbb{E}_\infty$-map of graded synthetic spectra

$$R^{\mathrm{Gr}} \to i_0 p_0 R \cong \nu\mathbb{S}(0).$$

We can further take the colimit on both sides, which is a symmetric monoidal functor, giving a $\mathbb{E}_\infty$-map of synthetic spectra

$$R \to \nu\mathbb{S}.$$

$\diamond$

Synthetic Obstruction Map

We will now use the map $R \to \nu\mathbb{S}$, to give a map of obstructions. Since R is an $\mathbb{E}_\infty$-ring, there is a symmetric monoidal structure on the module category $\mathrm{RMod}_R(\mathrm{Syn}_{\mathbb{F}_2})$.

Definition 5.16. Let ${}_{\tilde{4}}\nu\mathbb{S}$ be the cofiber of the map $\Sigma^{0,2}R \xrightarrow{\cdot x-\tilde{4}} R$ in $\mathrm{RMod}_R(\mathrm{Syn}_{\mathbb{F}_2})$. $\diamond$

Remark 5.17. The notation is supposed to suggest that ${}_{\tilde{4}}\nu\mathbb{S}$ has underlying synthetic spectrum $\nu\mathbb{S}$, where the free generator of R acts by multiplication by $\tilde{4}$. To see this, note we have a long exact sequence

$$\pi_{a,b-2}(R) \xhookrightarrow{x-\tilde{4}} \pi_{a,b}(R) \longrightarrow \pi_{a,b}({}_{\tilde{4}}\nu\mathbb{S}) \longrightarrow \pi_{a-1,b-1}(R) \xhookrightarrow{x-\tilde{4}} \pi_{a-1,b+1}(R)$$

and since $\pi_{a,b}(R) = \pi_{a,b}(\nu\mathbb{S})[x]$, we have

$$\pi_{a,b}({}_{\tilde{4}}\nu\mathbb{S}) = \pi_{a,b}(\nu\mathbb{S})[x]/(x-\tilde{4}) \cong \pi_{a,b}(\nu\mathbb{S})$$

so the map on underlying synthethic spectra

$$\nu\mathbb{S} \to R \to_{\tilde{4}} \nu\mathbb{S}$$

is an equivalence. $\diamond$

The $\mathbb{E}_\infty$-ring map $R \to \nu\mathbb{S}$ induces a symmetric monoidal map on module categories by the pushforward functor. Since $\nu\mathbb{S}$ is the unit in $\mathrm{Syn}_{\mathbb{F}_2}$, its module category $\mathrm{RMod}_{\nu\mathbb{S}}(\mathrm{Syn}_{\mathbb{F}_2})$ is equivalent to $\mathrm{Syn}_{\mathbb{F}_2}$.

Lemma 5.18. *The symmetric monoidal functor* $\mathrm{RMod}_R(\mathrm{Syn}_{\mathbb{F}_2}) \to \mathrm{Syn}_{\mathbb{F}_2}$ *induced by the map* $R \to \nu\mathbb{S}$*, sends* ${}_{\tilde{4}}\nu\mathbb{S}$ *to* $\nu\mathbb{S}/\tilde{4}$*.*

Proof. The pushforward is a left adjoint, so it commutes with colimits. We then have

$$\mathrm{cofib}_R(\Sigma^{0,2}R \xrightarrow{x-\tilde{4}} R) \otimes_R \nu\mathbb{S} \cong \mathrm{cofib}_{\mathrm{Syn}_{\mathbb{F}_2}}(\Sigma^{0,2}\nu\mathbb{S} \xrightarrow{\tilde{4}} \nu\mathbb{S}) \cong \nu\mathbb{S}/\tilde{4}.$$

$\square$

Combining this lemma with Proposition 5.1 and Example 5.2, we get

Proposition 5.19. *An $\mathbb{A}_{n-1}$-algebra structure on ${}_{\tilde{4}}\nu\mathbb{S}$ over R, induces a $\mathbb{A}_{n-1}$-algebra structure on $\nu\mathbb{S}/\tilde{4}$, and the obstruction for an $\mathbb{A}_n$-algebra structure on ${}_{\tilde{4}}\nu\mathbb{S}$ over R, gets mapped to the obstruction for an $\mathbb{A}_n$-algebra on $\nu\mathbb{S}/\tilde{4}$ by the map*

$$\begin{aligned} \pi_{2n-3,3}(\nu\mathbb{S}) &= \pi_0 \operatorname{Map}_R(\Sigma^{2n-3,3}R, {}_{\tilde{4}}\nu\mathbb{S}) \\ &\xrightarrow{\eta_M} \pi_0 \operatorname{Map}_R(\Sigma^{2n-3,3}R, \nu\mathbb{S}/\tilde{4}) \\ \xrightarrow{\Psi} \pi_0 \operatorname{Map}_{\mathrm{Syn}_{\mathbb{F}_2}}(\Sigma^{2n-3,3}\nu\mathbb{S}, \nu\mathbb{S}/\tilde{4}) &= \pi_{2n-3,3}(\nu\mathbb{S}/\tilde{4}). \end{aligned}$$

In particular, we have the following proposition.

Proposition 5.20. *If ${}_{\tilde{4}}\nu\mathbb{S}$ admits the structure of an $\mathbb{A}_4$-$\nu\mathbb{S}[\mathbb{S}^{0,2}]$-algebra structure then $\nu\mathbb{S}/\tilde{4}$ admits an $\mathbb{A}_5$-structure, implying that $\mathbb{S}/4$ admits an $\mathbb{A}_5$-algebra structure.*

Proof. In this case the $\mathbb{A}_5$-obstruction is well-defined for ${}_{\tilde{4}}\nu\mathbb{S}$, and as such the $\mathbb{A}_5$ obstruction for $\nu\mathbb{S}/\tilde{4}$ factors through the map

$$\pi_{7,3}(\nu\mathbb{S}) \to \pi_{7,3}(\nu\mathbb{S}/\tilde{4}).$$

We draw the homotopy groups of $\nu\mathbb{S}$ below, with vertical lines denoting $\tilde{2}$ multiplications. From this we see $\pi_{7,3}(\nu\mathbb{S})$ only consists of $\tilde{4}$-multiples so the map is null.

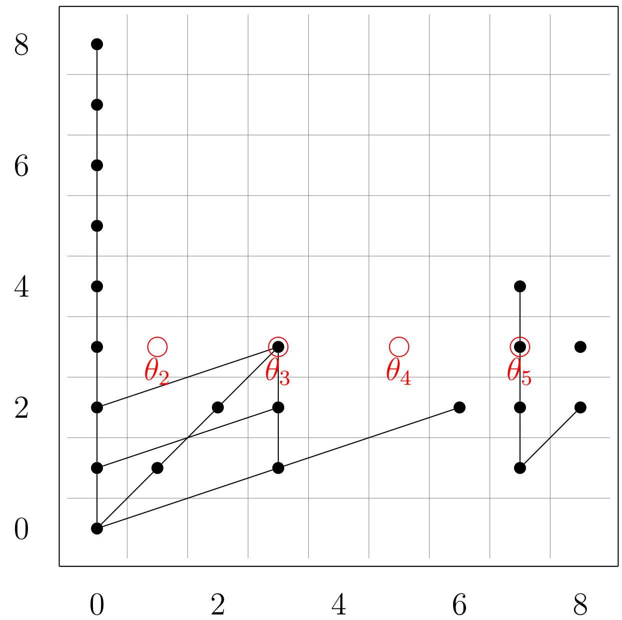


Figure 4: Bigraded homotopy groups of $\nu\mathbb{S}$ without τ-multiples. Each dot represent a copy of $\mathbb{F}_2$.

∎

Remark 5.21. The location of the obstruction for an $\mathbb{A}_4$-algebra structure on ${}_{\bar{4}}\nu\mathbb{S}$ over R is a null-group by the above picture so it suffices for Proposition 5.20 to show that ${}_{\bar{4}}\nu\mathbb{S}$ is an $\mathbb{A}_3$-algebra over R. ◇

We will now show ${}_{\bar{4}}\nu\mathbb{S}$ admits an $\mathbb{A}_3$-algebra structure.

6 Homotopy associative multiplications

In this section every stable ∞-category is 2-local, in the sense of Section 7.1. We will suppress the (2) from the notation.

Theorem 6.1. *Let $\mathcal{C}$ be a presentable symmetric monoidal stable ∞-category, $X \in \mathrm{Pic}(\mathcal{C})$ and $\varphi : X \to \mathbf{1}_{\mathcal{C}}$ be a map to the unit. If the map $Q_1(\varphi) : \Sigma X^{\otimes 2} \to \mathbf{1}_{\mathcal{C}}$ vanishes, then $1_{\mathcal{C}}/\varphi$ admits an $\mathbb{A}_3$-multiplication.*

To prove this result we will tackle the universal case.

Lemma 6.2. *There exist a symmetric monoidal stable locally graded presentable ∞-category $\mathcal{A} \in \mathrm{CAlg}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Gd}}\right)$, such that for any other $\mathcal{C} \in \mathrm{CAlg}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Gd}}\right)$, the diagram*

$$\begin{tikzcd} \mathrm{Fun}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L},\mathrm{Gd}}}(\mathcal{A}, \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d, "0"] \\ \mathrm{Map}_{\mathcal{C}}(T(1_{\mathcal{C}}), 1_{\mathcal{C}}) \arrow[r, "Q_1"'] & \mathrm{Map}_{\mathcal{C}}(\Sigma T^2(1_{\mathcal{C}}), 1_{\mathcal{C}}) \end{tikzcd}$$

is a pullback square, where Q_1 assigns the power operation Q_1 to maps in $T(1_{\mathcal{C}}) \to 1_{\mathcal{C}}$.

Proof. We will first construct $\mathcal{A}$. From Section 7.2, we have $\mathrm{Sp}^{\mathrm{Gr}}$ is the initial object in $\mathrm{CAlg}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Gd}}\right)$. Given $B \in \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{Gr}})$, from [Lur17, Remark 4.8.5.12] we get an adjunction

$$\mathrm{Fun}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Gd}}}\left(\mathrm{RMod}_B(\mathrm{Sp}^{\mathrm{Gr}}), \mathcal{D}\right) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp}^{\mathrm{Gr}})}\left(B, \mathrm{End}^{\mathrm{Gd}}(\mathcal{D})\right),$$

where $\mathrm{End}^{\mathrm{Gd}}$ is the graded algebra of endomorphisms of the unit described in [Lur15, Remark 2.4.9]. Let $\mathbb{S}\{v\}, \mathbb{S}\{w\} \in \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{Gr}})$ be the free $\mathbb{E}_\infty$-algebras on generators v, w, where v has degree 0 and filtration 1, and w has degree 1 and filtration 2. Let A be pushout in $\mathrm{Alg}_{\mathbb{E}_\infty}\left(\mathrm{Sp}^{\mathrm{Gr}}\right)$ of the diagram

$$\begin{tikzcd} \mathbb{S}\{w\} \arrow[r, "Q_1(v)\cdot"] \arrow[d, "0\cdot"'] & \mathbb{S}\{v\} \arrow[d] \\ \mathbb{S} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & A. \end{tikzcd}$$

From the above adjunction, we have

$$\begin{aligned} &\mathrm{Fun}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Gd}}}\left(\mathrm{RMod}_A(\mathrm{Sp}^{\mathrm{Gr}}), \mathcal{D}\right) \\ \simeq\ & \mathrm{Map}_{\mathrm{Sp}^{\mathrm{Gr}}}\left(\mathbb{S}(1), \mathrm{End}^{\mathrm{Gd}}(\mathcal{D})\right) \times_{\mathrm{Map}_{\mathrm{Sp}^{\mathrm{Gr}}}\left(\mathbb{S}^1(2), \mathrm{End}^{\mathrm{Gd}}(\mathcal{D})\right)} * \\ \simeq\ & \mathrm{Map}_{\mathcal{D}}(\mathbf{1}_{\mathcal{D}}(1), \mathbf{1}_{\mathcal{D}}) \times_{\mathrm{Map}_{\mathcal{D}}(\Sigma \mathbf{1}_{\mathcal{D}}(2), \mathbf{1}_{\mathcal{D}})} *. \end{aligned}$$

A symmetric monoidal functor in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Gd}}$ out of $\mathrm{RMod}_A(\mathrm{Sp}^{\mathrm{Gr}})$, is then given by a map $\varphi : \mathbf{1}_{\mathcal{D}}(1) \to \mathbf{1}_{\mathcal{D}}$, such that $Q_1(\varphi)$ is nulhomotopic. □

Remark 6.3. Applying the above result to the identity $\mathrm{RMod}_A(\mathrm{Sp}_{\mathrm{Gr}})$, we get a universal map is $A(1) \xrightarrow{\cdot v} A$, which is adjoint to the map $v : \mathbb{S}(1) \to A$. It then follows that a symmetric monoidal functor $F : \mathcal{A} \to \mathcal{D}$ determined by a map $\varphi : \mathbf{1}_{\mathcal{D}}(1) \to \mathbf{1}_{\mathcal{D}}$ sends v to φ, and A/v to $\mathbf{1}_{\mathcal{D}}/\varphi$. ◇

We will now study A more carefully.

Lemma 6.4. *The map given by $\mathbb{S}\{v\}(1) \xrightarrow{\cdot v} \mathbb{S}\{v\}$ induce an isomorphism*

$$\mathrm{Map}_{\mathrm{Sp}^{\mathrm{Gr}}}\left(\mathbb{S}^i(3), \mathbb{S}\{v\}(1)\right) \to \mathrm{Map}_{\mathrm{Sp}^{\mathrm{Gr}}}\left(\mathbb{S}^i(3), \mathbb{S}\{v\}\right)$$

Proof. By [Lur17, Example 3.1.3.14], the underlying graded spectrum of $\mathbf{1}(\mathbb{S}(1))$ is

$$\bigoplus_{n \geq 0} \mathrm{Sym}^n_{\mathrm{Sp}^{\mathrm{Gr}}}(\mathbb{S}(1)) \simeq \bigoplus_{n \geq 0} \mathrm{Sym}^n_{\mathrm{Sp}}(\mathbb{S})(n)$$

so we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{Sp}^{\mathrm{Gr}}}(\mathbb{S}^i(3), \mathbf{1}\{\mathbb{S}\}(1)) &\simeq \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}^i, \mathrm{Sym}^2_{\mathrm{Sp}}(\mathbb{S})) \\ \mathrm{Map}_{\mathrm{Sp}^{\mathrm{Gr}}}(\mathbb{S}^i(3), \mathbf{1}\{\mathbb{S}\}) &\simeq \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}^i, \mathrm{Sym}^3_{\mathrm{Sp}}(\mathbb{S})). \end{aligned}$$

The remaining part of this proof is to show that the map from $\mathrm{Sym}^2_{\mathrm{Sp}}(\mathbb{S}) = \Sigma^{\infty}_{+}\mathrm{B}\Sigma_{2(2)}$ to $\mathrm{Sym}^3_{\mathrm{Sp}}(\mathbb{S}) = \Sigma^{\infty}_{+}\mathrm{B}\Sigma_{2(2)}$ is an equivalence, where Σ_n is the symmetric group, and the map is induced from the inclusion of Σ_2 to Σ_3. Since the spectra are 2-local, it is enough to show the map induce an isomorphism on $\mathbb{F}_2$- homology. We have

$$H_*(\mathrm{B}\Sigma_2; \mathbb{F}_2) = H_*(\mathbb{R}\mathrm{P}^{\infty}; \mathbb{F}_2) = \mathbb{F}_2[x].$$

We can calculate $H_*(\mathrm{B}\Sigma_3; \mathbb{F}_2)$ using the Hochschild–Serre spectral sequence, with the normal subgroup $\mathbb{F}_3$

$$H_p(\mathbb{F}_2, H_q(\mathbb{F}_3, \mathbb{F}_2)) \implies H_{p+q}(\Sigma_3, \mathbb{F}_2).$$

We have

$$H_p(\mathbb{F}_3; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & p = 0 \\ 0 & \text{else} \end{cases}$$

so the groups in the spectral sequence are given by.

$$H_p(\mathbb{F}_2, H_q(\mathbb{F}_3, \mathbb{F}_2)) = \begin{cases} \mathbb{F}_2 & q = 0 \\ 0 & \text{else} \end{cases}$$

So the spectral sequence degenerates on the first page

$$H_*(\mathrm{B}\Sigma_3; \mathbb{F}_2) = H_*(\mathbb{R}\mathrm{P}^\infty; \mathbb{F}_2) = \mathbb{F}_2[x].$$

From this we get both spectra have the same $\mathbb{F}_2$-homology. We further have the composition

$$\Sigma^2 \hookrightarrow \Sigma^3 \xrightarrow{\text{sign}} \Sigma^2$$

is the identity, which implies the map $H_*(\mathrm{B}\Sigma_2; \mathbb{F}_2) \to H_*(\mathrm{B}\Sigma_3; \mathbb{F}_2)$ is injective. It follow then since they have isomorphic finite homology in each degree, that the spectra are equivalent. $\square$

Since our spectra are 2-local, we can also from the $\mathbb{F}_2$-homology, find a minimal cellular structure of them, displayed here.

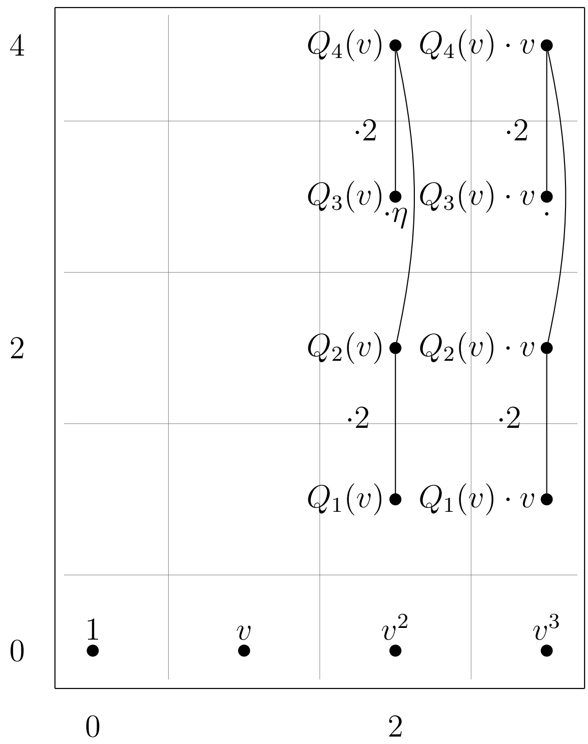


Figure 5: Cell Structure of $\mathbb{S}\{v\}$, in filtration 0 to 3, with horizontal axis giving filtration and vertical axis giving the degree. Each dot represent a copy of the sphere spectrum.

Lemma 6.5. *The map* $A(1) \xrightarrow{\cdot v} A$ *induce an isomorphism*

$$\mathrm{Map}_{\mathrm{Sp}^{\mathrm{Gr}}}(\mathbb{S}^i(3), A(1)) \to \mathrm{Map}_{\mathrm{Sp}^{\mathrm{Gr}}}(\mathbb{S}^i(3), A).$$

Proof. This follows from the previous lemma, as the only cells attached in the diagram

$$\begin{tikzcd}
\mathbb{S}\{w\} \arrow[r, "Q_1(v)\cdot"] \arrow[d, "0\cdot"'] & \mathbb{S}\{v\} \arrow[d] \\
\mathbb{S} \arrow[r] & A \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

in filtration 3 or less, is a cell $\mathbb{S}^1(2)$ denoted by w and a cell $\mathbb{S}^1(3)$ corresponding to $v \cdot w$, attached to $Q_1(v)$ and $Q_1(v) \cdot v$ respectively, so it has cell structure given in the following diagram.

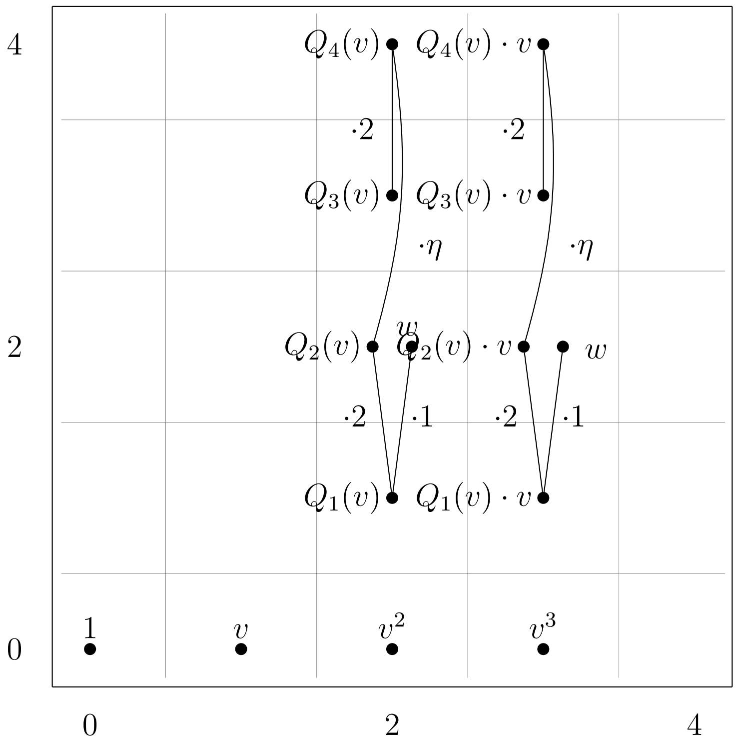
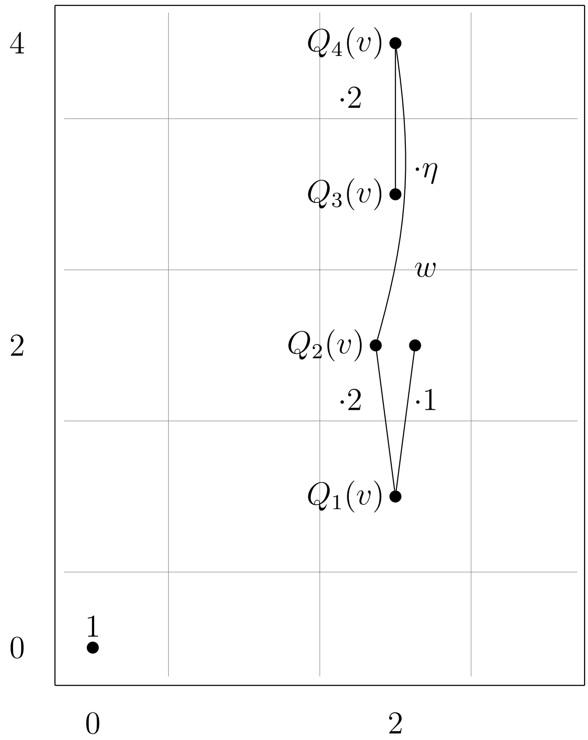


Figure 6: Cell Structure of A, in filtration 0 to 3, with horizontal axis giving filtration and vertical axis giving the degree. Each dot represent a copy of the sphere spectrum.

Which is identical in filtration 2 and 3. □

Proposition 6.6. *The cofiber* A/v *admits a* $\mathbb{A}_3$*-multiplication.*

Proof. From Lemma 6.5, we can calculate the cell structure of A/v.



This then shows there is no non-trivial maps from any spheres in filtration 3. In filtration 2, there are no maps from spheres of degree less than 2. The obstruction to giving a $\mathbb{A}_2$-structure on the cofiber of v lives in

$$[\mathbb{S}^1(2), A/v]_{\mathrm{Sp}^{\mathrm{Gr}}},$$

which is a nul-group so the obstruction vanishes automatically.

The obstruction to giving a $\mathbb{A}_3$-structure lives in

$$[(\Sigma A(1))^{\otimes 3}, A/v]_{\mathrm{RMod}_A(\mathrm{Sp}^{\mathrm{Gr}})} \simeq [\mathbb{S}^3(3), A/v]_{\mathrm{Sp}^{\mathrm{Gr}}}.$$

which is also a nulgroup, and so also vanishes. We can then conclude from Proposition 3.9, that A/v admits a $\mathbb{A}_3$-multiplication. □

Proof of Theorem 6.1. Let $\mathcal{C}$ be a presentable symmetric monoidal stable category, and let $X \in \mathrm{Pic}(\mathcal{C})$. Then the functor

$$\mathcal{C} \xrightarrow{\mathrm{id} \otimes X} \mathcal{C} \otimes \mathcal{C} \xrightarrow{m} \mathcal{C}$$

is an equivalence of $\mathcal{C}$, so $\mathcal{C}$ admits the structure of a locally graded ∞-category with $1_{\mathcal{C}}(1) = X$. A map $\varphi : X \to 1_{\mathcal{C}}$ with $Q_1(\varphi)$ nulhomotopic, then gives a symmetric monoidal functor $\mathrm{RMod}_A(\mathrm{Sp}^{\mathrm{Gr}}) \to \mathcal{C}$ sending A/v to $1_{\mathcal{C}}/\varphi$ by Lemma 6.2. Since A/v admits an $\mathbb{A}_3$-algebra structure, the functor induces a $\mathbb{A}_3$-algebra structure on $1_{\mathcal{C}}/\varphi$. □

We can apply this result to finish our study of $\mathbb{S}/4$. From Proposition 5.20, to prove $\mathbb{S}/4$ admits an $\mathbb{A}_5$-multiplication, we just have to prove that ${}_{\tilde{4}}\nu\mathbb{S}$ admits a $\mathbb{A}_3$-multiplication.

Lemma 6.7. *Given a symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ and a $\mathbb{E}_\infty$-algebra $X \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C})$ and a map $\varphi : A \to X$, we have*

$$F(Q_1(\varphi)) = Q_1(F(\varphi)).$$

Proposition 6.8. *the synthetic module ${}_{\tilde{4}}\nu\mathbb{S}$ admits an $\mathbb{A}_3$-algebra structure over R.*

Proof. The module ${}_{\tilde{4}}\nu\mathbb{S}$ is the cofiber of the map $x - \tilde{4} : \Sigma^{0,2}R \to R$ in $\mathrm{RMod}_R(\mathrm{Syn}_{\mathbb{F}_2})$. $\Sigma^{0,2}R$ is a unit in $\mathrm{RMod}_R(\mathrm{Syn}_{\mathbb{F}_2})$, so we are left to check $Q_1(x - \tilde{4})$ is nulhomotopic.

We have a diagram

$$\begin{tikzcd}
{}_R[\Sigma^{1,3}R, R] \arrow[r, "\tau^{-1}"'] \arrow[d, "\simeq"] & {}_{\mathbb{S}[x]}[\Sigma\mathbb{S}[x], \mathbb{S}[x]] \arrow[d, "\simeq"] \\
{}_{\mathrm{Syn}_{\mathbb{F}_2}}[\Sigma^{1,3}\nu\mathbb{S}, R] \arrow[r, "\tau^{-1}"] & {}_{\mathrm{Sp}}[\Sigma\mathbb{S}, \mathbb{S}[x]],
\end{tikzcd}$$

where the horizontal maps are injections, since there is no τ-torsion in these ranges. To check $Q_1(x - \tilde{4})$ is nulhomotopic, we can then equally show $Q_1(x - 4)$ in Sp is nulhomotopic. Here we have the rule

$$Q_1(x - 4) = 16Q_1(x) + x^2Q_1(4) + \eta x4.$$

We have $Q_1(4) = 0$ and η is 2-torsion, so the last two terms vanish.

For $Q_1(x)$, recall it is given as the composition

$$\mathbb{S}^1 \to D_2(\mathbb{S}) \xrightarrow{D_2(x)} D_2(\mathbb{S}[x]) \to \mathbb{S}[x]$$

where $D_2(X) = X^{\otimes 2}_{h\Sigma_2}$ is the second extended power. The multiplication on $\mathbb{S}[x] \simeq \coprod_{n\geq 0}\mathbb{S}$ is given by

$$\left(\coprod_{n\geq 0}\mathbb{S}\right) \otimes \left(\coprod_{m\geq 0}\mathbb{S}\right) \simeq \coprod_{n,m\geq 0} \mathbb{S}\otimes\mathbb{S} \xrightarrow{m} \coprod_{n\geq 0}\mathbb{S}.$$

which sends the (n, m)'th term to the $n + m$'th term with the multiplication map of $\mathbb{S}$. From this we get a commuting square

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r] & D_2(\mathbb{S}) \arrow[r, "D_2(x)"] \arrow[d, "m"'] & D_2(\mathbb{S}[x]) \arrow[d, "m"] \\
 & \mathbb{S} \arrow[r, "x^2"] & \mathbb{S}[x].
\end{tikzcd}$$

We then see that $Q_1(x) = Q_1(1)x^2 = 0$, Since $Q_1(1) = 0$ in $\mathbb{S}$.

□

Corollary 6.9. *The spectrum* $\mathbb{S}/4$ *admits an* $\mathbb{A}_5$*-multiplication.*

7 Appendix

7.1 Localising Stable categories at primes

Since our primary object of interest is $\mathbb{S}/4$, we would like to focus on 2-complete presentable categories. For this we introduce Bousfield localisation.

Definition 7.1. A map $f : X \to Y$ in Sp is an *E-equivalence*, If $E \otimes f : E \otimes X \to E \otimes Y$ is an equivalence of spectra. A spectrum Z is *E-local* if for every E-equivalence $f : X \to Y$,

$$\mathrm{Map}(Y, Z) \xrightarrow{f_*} \mathrm{Map}(X, Z)$$

is an equivalence. An *E-localization* of X is an E-local spectrum $L_E X$, with an E-equivalence $X \to L_E(X)$. ◇

An E-localization always exists and is unique.

Example 7.2. Localization with respect to the Moore spectra $E = S\mathbb{Z}_{(p)}$ is called p-localization. The spectrum $\mathbb{S}/4$ is 2-local. ◇

We denote the full ∞-subcategory of Sp spanned by p-local spectra by $\mathrm{Sp}_{(p)}$. By [Lur17, Proposition 2.2.1.9], the functor p-localization is symmetric monoidal, and the inclusion is lax symmetric monoidal. Since lax symmetric monoidal functors induce functors on algebra categories, we get a functor

$$\mathrm{Alg}_{\mathbb{A}_\mathrm{n}}(\mathrm{Sp}_{(p)}) \to \mathrm{Alg}_{\mathbb{A}_\mathrm{n}}(\mathrm{Sp}).$$

Therefore to prove that $\mathbb{S}/4$ admits an $\mathbb{A}_\mathrm{n}$-algebra structure, it is enough to prove it is an $\mathbb{A}_\mathrm{n}$-algebra in $\mathrm{Sp}_{(p)}$, as the cofiber of $4 : \mathbb{S}_{(2)} \to \mathbb{S}_{(2)}$, where $\mathbb{S}_{(2)}$ is the 2-local sphere.

Remark 7.3. We have the pair $(\mathrm{Sp}_{(2)}, \mathbb{S}_{(2)})$ is idempotent in $\mathcal{P}\mathrm{r}^\mathrm{L}$ in the sense of [Lur17, page 720], so we have the forgetful functor $\mathrm{Mod}_{\mathrm{Sp}_{(2)}}(\mathcal{P}\mathrm{r}^\mathrm{L}) \to \mathcal{P}\mathrm{r}^\mathrm{L}$ determines a fully faithful embedding, whose left adjoint is given by the tensoring with $\mathrm{Sp}_{(2)}$ in $\mathcal{P}\mathrm{r}^\mathrm{L}$.

◇

Definition 7.4. We define the ∞-category of presentable stable 2-local categories $\mathcal{P}\mathrm{r}^\mathrm{L}_{(2)}$ as $\mathrm{Mod}_{\mathrm{Sp}_{(2)}}(\mathcal{P}\mathrm{r}^\mathrm{L})$. ◇

We note a lemma we need for Section 6.

Lemma 7.5. *We have an equivalence in* $\mathcal{P}\mathrm{r}^\mathrm{L}_{(2)}$

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}) \otimes \mathrm{Sp}_{(2)} \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}_{(2)}).$$

Proof. Using [Lur17, Proposition 4.8.1.17], we have

$$\begin{aligned}
\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}) \otimes \mathrm{Sp}_{(2)} &\simeq \mathrm{RFun}(\mathrm{Fun}(\mathcal{C}, \mathrm{Sp})^{\mathrm{op}}, \mathrm{Sp}_{(2)}) \\
&\simeq \mathrm{LFun}(\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}), \mathrm{Sp}_{(2)}^{\mathrm{op}})^{\mathrm{op}} \\
&\simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}_{(2)}^{\mathrm{op}})^{\mathrm{op}} \\
&\simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}_{(2)})
\end{aligned}$$

□

In particular, we have $\mathrm{Sp}^{\mathrm{Gr}} \otimes \mathrm{Sp}_{(2)} \simeq \mathrm{Sp}_{(2)}^{\mathrm{Gr}}$.

7.2 Locally Graded stable ∞-categories

We recall the notion of locally graded stable ∞-category from [Lur15], and adapt to the setting of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Definition 7.6. Let $\mathcal{C}$ be a stable ∞-category. A local grading of $\mathcal{C}$ is an equivalence from $\mathcal{C}$ to itself. We will use the term locally graded ∞-category to refer to a pair $(\mathcal{C}, T)$, where T is a local grading on $\mathcal{C}$. ◇

Example 7.7. The stable ∞-category $\mathrm{Sp}^{\mathrm{Gr}}$ admits a local grading by the shift map $X \mapsto X(1)$, which increases the indices by 1. We $\mathrm{Sp}^{\mathrm{Gr,fin}}$ denote the full subcategory spanned by graded spectrum satisfying:

1. The spectrum X_i is finite for each index.
2. For all but finitely many indices, the spectrum X_i vanishes.

◇

Remark 7.8. From [Lur15, Corollary 2.4.4], the data of a local grading T on a stable ∞-category $\mathcal{C}$ is equivalent to a monoidal functor

$$\begin{aligned}
\mathbb{Z}^{\mathrm{ds}} &\to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C}) \\
\mathbb{S}(1) &\mapsto T.
\end{aligned}$$

◇

We will now handle the case where $\mathcal{C}$ is presentable.

Lemma 7.9. *Let $\mathcal{C}$ be a presentable stable ∞-category. Then the data of a local grading on $\mathcal{C}$ is equivalent to a monoidal colimit preserving functor*

$$\begin{aligned}
\mathrm{Sp}^{\mathrm{Gr}} &\to \mathrm{LFun}(\mathcal{C}, \mathcal{C}) \\
\mathbb{S}(1) &\mapsto T.
\end{aligned}$$

Proof. From the previous remark, we know a local grading is equivalent to a object in $\mathrm{Fun}^{\otimes}\left(\mathbb{Z}^{\mathrm{ds}}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{C})\right)$. Since such a functor on objects sends every n to a equivalence, the image must lie in the full subcategory of colimit preserving functors $\mathrm{LFun}(\mathcal{C},\mathcal{C}) \in \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$. Since the stable Yoneda embedding $\mathrm{Fun}((-)^{\mathrm{op}}, \mathrm{Sp}) : \mathcal{C}\mathrm{at}_{\infty} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ is symmetric monoidal and left adjoint to the inclusion, we have

$$\mathrm{Fun}^{\otimes}\left(\mathbb{Z}^{\mathrm{ds}}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{C})\right) \simeq \mathrm{Fun}^{\otimes}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}}\left(\mathrm{Sp}^{\mathrm{Gr}}, \mathrm{LFun}(\mathcal{C},\mathcal{C})\right)$$

Using that $\mathbb{Z}^{\mathrm{ds}} \simeq (\mathbb{Z}^{\mathrm{ds}})^{\mathrm{op}}$. □

Definition 7.10. The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Gd}}$ of locally graded stable presentable ∞-categories is defined as $\mathrm{LMod}_{\mathrm{Sp}^{\mathrm{Gd}}}\left(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}\right)$. ◇

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